Evidence in favor of the Baez-Duarte criterion for the Riemann Hypothesis

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Abstract

We presents results of the numerical experiments in favor of the Baez-Duarte criterion for the Riemann Hypothesis. We give formulae allowing calculation of numerical values of the numbers c_k appearing in this criterion for arbitrary large k. We present plots of c_k for $k \in (1, 10^9)$.

1. Introduction.

In 1997 K. Maślanka [1] proposed a new formula for the zeta Riemann function valid on the whole complex plane $\mathbb{C}$ except a point $s = 1$:

$$\zeta(s) = \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{\Gamma(k+1-\frac{s}{2})}{\Gamma(1-\frac{s}{2})}\frac{A_k}{k!}$$

where coefficients A_k are given by

$$A_k = \sum_{j=0}^{k}\binom{k}{j}(2j-1)\zeta(2j+2).$$

This formula was rigorously proved by L. Baez-Duarte in 2003 [2]. In the subsequent preprint [3] the same author proved the new criterion for the Riemann Hypothesis, the journal version of it appeared two years later [4]. The Riemann

Hypothesis (RH) states that the nontrivial zeros ρ of the function $\zeta(s)$ have the real part equal $\Re(\rho)=\frac{1}{2}$. Although Riemann did not request it, today it is often demanded additionally that zeros on the critical line $\Re(s)=\frac{1}{2}$ should be simple. Baez-Duarte considered the sequence of numbers c_k defined by:

$$c_k=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{\zeta(2j+2)}. \tag{1}$$

He proved that RH is equivalent to the following rate of decreasing to zero of the above sequence:

$$c_k=\mathcal{O}(k^{-\frac{3}{4}+\epsilon}) \qquad \text{for each} \quad \epsilon>0. \tag{2}$$

Furthermore, if ϵ can be put zero, i.e. if $c_k=\mathcal{O}(k^{-\frac{3}{4}})$, then the zeros of $\zeta(s)$ are simply. Baez-Duarte also proved in [4] that it is not possible to replace $\frac{3}{4}$ by $\frac{3}{4}+\epsilon$.

Neither in [4] nor in [6] it is explicitly written whether the sequence c_k starts from $k=0$ or $k=1$. However in [4] a few formulas contain $k=0$, i.e. summation starts from c_0. The point is that if we allow $k=0$, for which $c_0=6/\pi^2$, then the inversion formula (see e.g. [7]) is fulfilled:

$$\frac{1}{\zeta(2k+2)}=\sum_{j=0}^{k}(-1)^j\binom{k}{j}c_j. \tag{3}$$

However I do not see application of the above formula, except the possibility of checking some of the statements made in [13]. Furthermore, if the Baez-Duarte sequence c_k starts from $k=0$ then the following identity holds:

$$\sum_{k=0}^{\infty}\frac{c_k x^k}{k!}=e^x\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!\zeta(2k+2)}. \tag{4}$$

It is an application of the general formal identity:

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}a_j\right)\frac{x^k}{k!}=e^x\sum_{k=0}^{\infty}\frac{a_k x^k}{k!}, \tag{5}$$

where a_k should not increase to fast with k to ensure convergence of series. [1] Putting here $a_j=(-1)^j b_j$ gives the usual formula appearing in the finite difference theory (see [14] §1):

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^j\binom{k}{j}b_j\right)\frac{x^k}{k!}=e^x\sum_{k=0}^{\infty}\frac{(-1)^k b_k x^k}{k!}. \tag{6}$$

[1] Indeed, collecting on the l.h.s. terms multiplying a_j we get: $a_j\sum_{k=j}^{\infty}\binom{k}{j}\frac{x^k}{k!}=a_j\sum_{k=j}^{\infty}\frac{k!}{j!(j-k)!}\frac{x^k}{k!}=a_j\frac{x^j}{j!}\sum_{n=0}^{\infty}\frac{x^n}{n!}=a_j\frac{x^j}{j!}e^x$ and summing over j gives r.h.s.

The identity (4) can be used to establish the connection with the Riesz criterion for RH (original paper [9], discussed in [4], [11]). Riesz has considered the function:

$$R(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k!\zeta(2k+2)}.$$

Unconditionally it can be proved that $R(x) = \mathcal{O}(x^{1/2+\epsilon})$, see [8] §14.32. Riesz has proved that the Riemann Hypothesis is equivalent to slower increasing of the function $R(x)$:

$$RH \Leftrightarrow R(x) = \mathcal{O}\left(x^{1/4+\epsilon}\right). \tag{7}$$

But from (4) we get:

$$\sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = \frac{e^x}{x} R(x) \tag{8}$$

thus the generating function for c_k can be expressed by $R(x)$. In [10] it is proved, that for any real number $\delta > -3/2$ we have

$$R(x) = O(x^{\delta+1}) \Leftrightarrow c_k = O(k^{\delta}). \tag{9}$$

Proof is based on the relation $R(k)/k \approx c_k$.

2. Computer experiments

The criterion (2) seemed to be very well suited for the computer verification. At the end of [3] Baez-Duarte wrote a sentence “A test for the first c_k up to $k = 1000$ shows a very pleasant smooth curve”. However for larger values of k the true behavior of the sequence turned out to be more complicated: instead of monotonic tending to zero there appeared oscillations and c_k changed the sign at first for[2] $k = 19320$: $c_{19319} = -1.7870567 \times 10^{-13}$ while $c_{19320} = 9.170232808\ldots \times 10^{-12}$. The next sign change is: $c_{22526} = 2.2292905301\ldots \times 10^{-13}$ but $c_{22527} = -6.5057526 \times 10^{-12}$.

To my knowledge the first plot of c_k for k up to 95000 appeared in the book [5] published in Polish. The same plot was reproduced in [4]. Data used to make this plot consisted of c_k calculated every 500-th k — it is very time consuming to get c_k directly from (1). Indeed, for large j the values of $\zeta(2j+2)$ very quickly become practically equal to 1, thus the summation of alternating series gives wrong result when not performed with sufficient number of digits accuracy. For example, the Table I below presents values of the partial sums for c_{12000} recorded every thousand summands (the calculation was performed with precision of 9000 digits).

Let us remark that the partial sums for n and $12000-n$ are of the same order. The binomial coefficients become very large numbers in the middle and to get accurate

[2] in fact if the sequence c_k starts from $k = 0$ the first sign change occurs for $c_0 = 6/\pi^2 > 0$ and $c_1 = 6/\pi^2 - 90/\pi^4 = (6\pi^2 - 90)/\pi^4 \approx (-30/\pi^4) < 0$ (more precisely $c_1 = -0.3160113011$...)

value of c_k one needs a lot of digits accuracy during the calculation. Maślanka has used `Mathematica` to perform these calculation. Over three years ago I started to calculate c_k using the free package PARI/GP [12] developed especially for number theoretical purposes and which allows practically arbitrary accuracy arithmetics both fixed-point as well as floating-point. I started to calculate consecutive c_k for each k with the help of the following script in Pari:

```
\p 3500    /*     precision set to 3500 digits    */
allocatemem(250000000)
range=10000     /*     the largest subscript in c_k         */
denomin=vector(range);
for (n=1, range, denomin[n]=zeta(2*n));
default(format, "e22.20")
{
for  (k=1, range, c=sum(j=0,k,((-1)^j)*binomial(k,j)/denomin[j+1]);
write("c_k.dat",k," ", c))
}
```

Table I

n	$\sum_{j=0}^{n} (-1)^j \binom{12000}{j} \frac{1}{\zeta(2j+2)}$
1000	$8.6575528427959311728\times 10^{1492}$
2000	$1.0610772171540382076\times 10^{2346}$
3000	$2.6820721693716011525\times 10^{2928}$
4000	$8.4511383022435967124\times 10^{3314}$
5000	$1.8751018390471552047\times 10^{3537}$
6000	$8.3417729099514988532\times 10^{3609}$
7000	$1.3393584564622537177\times 10^{3537}$
8000	$4.2255691511217983562\times 10^{3314}$
9000	$8.9402405645720038417\times 10^{2927}$
10000	$2.1221544343080764152\times 10^{2345}$
11000	$7.8705025843599374298\times 10^{1491}$
12000	$-1.6973092190852083930\times 10^{-7}$

The problem I have encountered during these calculations was that it seems to be not possible to change accuracy of calculation during running the script (the command `\p 3500` above). Thus I had to change the precision by hand. It turned out that when the precision was to small produced values of c_k were obviously wrong, something like ten to the very large power. The rule learned from these examples for precision set to make calculations confident was that the number of digits should be at least enough to distinguish between 1 and $1 + \frac{1}{2^k}$ in the zeta appearing in

(1), i.e. the precision set to calculate c_k should be at least `\p` $= k * \log_{10}(2)$. Table II presents the real example I have met during calculations: when the precision was set to 60000 digits the values of c_k for k between 198000 and 200000 were $(198000 \times \log_{10}(2) = 59603.93914, \quad 200000 \times \log_{10}(2) = 60205.99913)$:

Table II

k	c_k
198000	-8.1809420017968747912 $\times 10^{-9}$
198500	-8.1130397250007379108$\times 10^{-9}$
199000	-8.0431163120575296823$\times 10^{-9}$
199500	3.4122583912205353616 $\times 10^{49}$
200000	-1.9276608381598523688$\times 10^{200}$

Maślanka kindly send me values of c_k from his calculations up to $k = 95000$ with k jumping in intervals of 500, i.e. $k = 500l$. Autumn 2005 I have started to continue this efforts on the cluster of 8 processors Xeon 2.8 GHz, with 4 GB RAM per node of two processors[3], with the aim to reach $k = 200000$ also every 500-th value of k using PARI/GP computer algebra system [12]. During last five months of computations between 4 and 6 processors I have used to calculate c_k in different intervals of k. When these calculations were running I have learned of the paper [4] where explicit formulae for c_k in terms of zeros of $\zeta(s)$ were given. Quite recently there appeared the paper [6] where the prescription to obtain c_k very quickly were also given. In view of these developments there is no need to continue very time consuming calculations based on the formula (1). The only benefit of these calculation was the possibility to compare c_k obtained by means of formulae presented in [4] and [6] against those c_k obtained from the generic formula (1). It should be stressed that calculations based on (1) does not assume the validity of Riemann Hypothesis in contrast to formulae presented by Maślanka or below. Using these formulae c_k can be calculated very quickly for practically arbitrary k — it is very time consuming to calculate c_k without assuming RH.

3. Explicit formulae.

The formulae presented in [4] and in [6] expressing c_k directly in terms of the zeros of $\zeta(s)$ are essentially the same, they differ in the manner they were derived. Maślanka has used the binomial transforms discussed in [14] while Baez-Duarte is developing the whole machinery by himself. The formulae of these two authors can be written as a sum of two parts: quickly decreasing with k trend $\bar{c}_k$ and oscillations $\tilde{c}_k$:

[3] because I have used 32-bits version of PARI/GP I was able to use 2^{31} bytes =2 GB of RAM per process

$$c_k = \bar{c}_k + \tilde{c}_k$$

where:

$$\bar{c}_k = -\frac{1}{(2\pi)^2}\sum_{m=2}^{\infty}\frac{B(k+1,m)}{\Gamma(2m-1)}\frac{(-1)^m(2\pi)^{2m}}{\zeta(2m-1)} \tag{10}$$

and oscillating part:

$$\tilde{c}_k = \sum_{\rho}\frac{\Gamma(k+1)\Gamma(\frac{1+\rho}{2})}{\Gamma(k+1+\frac{1+\rho}{2})}\frac{1}{\zeta'(1-\rho)} = \sum_{\rho} B\left(k+1, \frac{1+\rho}{2}\right)\frac{1}{\zeta'(1-\rho)} \tag{11}$$

where it is assumed that zeros of $\zeta(s)$ are simple: $\zeta'(\rho) \neq 0$ and the sum is over all (i.e. on the positive as well as negative imaginary axis) nontrivial zeros of the $\zeta(s)$, i.e. $\zeta(\rho) = 0$ and $\Im\rho \neq 0$ and

$$B(w,z) = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)}$$

is the Beta function. In fact Baez-Duarte is skipping the trend remarking only that it is of the order $o(1/k)$ (Remark 1.6 in [4]). Theoretically the formula for $\tilde{c}_k$ is valid in the limit of large k, but surprisingly the numbers produced from the above formulae (10) and (11) are practically the same as obtained from the generic formula (1) for all k, e.g. already for $k = 2$ we get $c_2 = -0.25699711$ from (1), while (10) and (11) give $c_2 = -0.256969863$ and accuracy increases with k. It suggests that the integrals J_k appearing in [4] in the proof of the Theorem 1.5 are decreasing to zero rather fast with k.

First let us consider trend. It can be calculated directly from (10):

$$\bar{c}_k = -\frac{1}{(2\pi)^2}\sum_{m=2}^{\infty}\frac{1}{(k+1)(k+2)\ldots(k+m)m(m+1)\ldots(2m-2)}\frac{(-1)^m(2\pi)^{2m}}{\zeta(2m-1)} \tag{12}$$

Table III

k	$\bar{c}_k$ from eq.(12)	$\bar{c}_k$ from eq.(13)	$\bar{c}_k$ from eq.(14)
1	$-2.60052406393 \times 10^{-1}$	$-4.0752814729 \times 10^{3}$	$-1.6421193331 \times 10^{1}$
10	$-6.9069591105 \times 10^{-2}$	$-2.0455052855 \times 10^{1}$	$-1.6421193331 \times 10^{-1}$
10^2	$-1.4804264464 \times 10^{-3}$	$-5.9943727867 \times 10^{-3}$	$-1.6421193331 \times 10^{-3}$
10^3	$-1.6248041420 \times 10^{-5}$	$-1.6824923096 \times 10^{-5}$	$-1.6421193331 \times 10^{-5}$
10^4	$-1.6403755367 \times 10^{-7}$	$-1.6418022398 \times 10^{-7}$	$-1.6421193331 \times 10^{-7}$
10^5	$-1.6419448299 \times 10^{-9}$	$-1.6420436474 \times 10^{-9}$	$-1.6421193331 \times 10^{-9}$
10^6	$-1.6421018816 \times 10^{-11}$	$-1.6421113244 \times 10^{-11}$	$-1.6421193331 \times 10^{-11}$
10^7	$-1.6421175880 \times 10^{-13}$	$-1.6421185279 \times 10^{-13}$	$-1.6421193331 \times 10^{-13}$
10^8	$-1.6421191586 \times 10^{-15}$	$-1.6421192526 \times 10^{-15}$	$-1.6421193331 \times 10^{-15}$
10^9	$-1.6421193157 \times 10^{-17}$	$-1.6421193251 \times 10^{-17}$	$-1.6421193331 \times 10^{-17}$

Using this formula I was able to produce every 500-th value of $\bar{c}_k$ for $k = 500, 1000, \ldots 10^9$ performing calculations in Pari with 100 digits accuracy in about 4 hours. For large k I have used following asymptotic expansion of (12):

$$\bar{c}_k = -\frac{1}{(k+1)(k+2)}\left(\frac{(2\pi)^2}{2\zeta(3)} - \frac{(2\pi)^4}{12(k+3)\zeta(5)} + \frac{(2\pi)^6}{120(k+3)(k+4)\zeta(7)}\right). \quad (13)$$

It can be further simplified to:

$$\bar{c}_k = -\frac{1}{k^2}\frac{(2\pi)^2}{2\zeta(3)} \quad (14)$$

The comparison of these formulae is given in Table III.

Now we consider the oscillating part $\tilde{c}_k$. Since PARI/GP does not have built in $B(x,y)$ function, I had to use $\Gamma(z)$ functions instead. Because of the fast growth of the $\Gamma(x)$ function even in PARI/GP it was not possible to pursue with formula (11) for large k. Namely it crashes for $k = 356000$ because of overflow. But there is a following asymptotic formula (see e.g. [15], §1.8.7):

$$\frac{\Gamma(x)}{\Gamma(x+a)} \sim x^{-a}, \qquad x \to \infty$$

thus we have

$$B(a,x) \sim x^{-a}\Gamma(a) \qquad \text{for } \; x \; \text{ large.} \quad (15)$$

Using it for large k and assuming the Riemann Hypothesis: $\rho_l = \frac{1}{2} + i\gamma_l$, $\bar{\rho}_l = \frac{1}{2} - i\gamma_l (= 1 - \rho_l)$ after collecting together in pairs conjugate zeros we get:

$$\tilde{c}_k = \frac{2}{(k+1)^{\frac{3}{4}}}\sum_{l=1}^{\infty} \alpha_l \cos\left(\frac{1}{2}\gamma_l \log(k+1)\right) - \beta_l \sin\left(\frac{1}{2}\gamma_l \log(k+1)\right), \quad (16)$$

where I have denoted:

$$\alpha_l = \Re\left(\frac{\Gamma(\frac{1+\rho_l}{2})}{\zeta'(\bar{\rho}_l)}\right), \quad (17)$$

$$\beta_l = \Im\left(\frac{\Gamma(\frac{1+\rho_l}{2})}{\zeta'(\bar{\rho}_l)}\right). \quad (18)$$

In (16) the decreasing of c_k like $k^{-\frac{3}{4}}$ is obtained as an overall amplitude of the "waves" composed of the cosines and sines with the "frequencies" proportional to imaginary parts of the nontrivial zeros of $\zeta(s)$. The coefficients α_l and β_l decrease to zero very fast with l. Namely using the Hadamard product for $\zeta(s)$:

$$\zeta(s) = \frac{(2\pi)^s e^{-(1+C/2)s}}{2(s-1)\Gamma(s/2+1)}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

where $C = 0.57721566490153286\ldots$ is the Euler constant, the derivative of $\zeta(s)$ at zeros can be computed. Taking into account miraculous simplifications, $\rho\rho_l + \bar{\rho}\rho_l = \rho_l$ and the identity

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$$

I have obtained that:

$$|\alpha_l| \propto e^{-\pi\gamma_l/4}, \qquad |\beta_l| \propto e^{-\pi\gamma_l/4}. \tag{19}$$

Because imaginary parts of zeros take large values it suffices to sum in (16) over a few first zeros. I have used 10 zeros and the table below gives coefficients α_l and β_l and comparison with (19).

Table IV

l	α_l	β_l	$e^{-\pi\gamma_l/4}$
1	2.029173866 $\times 10^{-5}$	$-3.315924256 \times 10^{-5}$	1.50914×10^{-5}
2	-3.333265938 $\times 10^{-8}$	$-1.298336420 \times 10^{-7}$	6.75315×10^{-8}
3	2.886139424 $\times 10^{-9}$	$-4.153918097 \times 10^{-9}$	2.94404×10^{-9}
4	4.813880001 $\times 10^{-11}$	$-6.332017430 \times 10^{-11}$	4.19039×10^{-11}
5	7.546769513 $\times 10^{-12}$	$7.526891498 \times 10^{-12}$	5.83506×10^{-12}
6	6.162524600 $\times 10^{-14}$	$1.942118979 \times 10^{-13}$	1.51209×10^{-13}
7	-1.578482027 $\times 10^{-14}$	$1.184829593 \times 10^{-14}$	1.10374×10^{-14}
8	-1.07138189 2$\times 10^{-15}$	$-2.209146437 \times 10^{-15}$	1.66491×10^{-15}
9	9.328038737 $\times 10^{-19}$	$-7.472197226 \times 10^{-17}$	4.22403×10^{-17}
10	1.747829093 $\times 10^{-17}$	$1.122667624 \times 10^{-17}$	1.05303×10^{-17}

However already calculations with the first zero $\gamma_1 = 14.13472514173469\ldots$ give numbers which differ much less than 1% (see $l = 1$ and $l = 2$ in the above table) from those calculated with larger number of terms in (16) as well as with c_k for $k < 200000$ obtained directly from (1) without assuming RH. The plots of c_k for k up to 10^9 obtained from these formulae are given in the Fig.1 and Fig.2. In Fig.2 there is logarithmic k-axis and thus the plot has a constant "wavelength", not depending on k like on the Fig.1. The envelope is given by:

$$y = \frac{2A}{(k+1)^{3/4}}\ , \qquad A = 0.777506276445256 \times 10^{-5} \tag{20}$$

and was obtained in the following way: First I have maintained in (16) only the first zero $\rho_1 = \frac{1}{2} + \gamma_1$:

$$\tilde{c}_k = \frac{2}{(k+1)^{\frac{3}{4}}}\left(\alpha_1 \cos\left(\frac{1}{2}\gamma_1 \log(k+1)\right) - \beta_1 \sin\left(\frac{1}{2}\gamma_1 \log(k+1)\right)\right). \tag{21}$$

Next I made use of the identity:

$$a\cos(\theta) - b\sin(\theta) = A\sin(\phi - \theta) \quad \text{where} \quad A = \sqrt{a^2+b^2}, \quad \phi = \arctan\left(\frac{a}{b}\right) \tag{22}$$

to obtain:

$$\tilde{c}_k = \frac{2}{(k+1)^{\frac{3}{4}}}\sqrt{\alpha_1^2+\beta_1^2}\sin\left(\phi - \frac{1}{2}\gamma_1\log(k+1)\right). \tag{23}$$

from which (20) follows and numerical value of A is obtained from α_1 and β_1 in Table IV. Here$\phi = -0.54916 (= 56.497°)$ Let us remark that this value of A agrees very well with amplitude reported by Beltraminelli and Merlini [16]. It is interesting to note that lhs of the above formula is valid not only for integer k but also for real k, thus using the approximation $c_k \approx R(k)/k$ derived in [10] we can write for large x:

$$R(x) = 2x^{1/4}\left(\alpha_1\cos\left(\frac{1}{2}\gamma_1\log(x)\right) - \beta_1\sin\left(\frac{1}{2}\gamma_1\log(x)\right)\right) \tag{24}$$

There is another way of checking accuracy of the above equation (23). Namely assuming that (23) is true and denoting by $k^{'}$ and $k^{''}$ two consecutive zeros of $c_{k'} = 0, c_{k''} = 0$ we get for γ_1

$$\gamma_1 = \frac{2\pi}{\log((k''+1)/(k'+1))} \tag{25}$$

To make sense, in the latter approach an independent of (23) and relatively fast method of calculating c_k is needed. In fact in [4] Baez-Duarte gives among others following formula being the transformation of (1):

$$c_k = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\left(1-\frac{1}{n^2}\right)^k. \tag{26}$$

Using this expression I have searched numerically for sign changes of c_k up to $k = 10^9$ and Table V presents γ_1 calculated from the above formula (25) together with consecutive zeros extrapolated from integer values using the linear approximation (let us remind that $\sin(x)$ has derivative ± 1 at zeros!). The correct value is $\gamma_1 = 14.134725141734693790457 25\ldots$.

In Fig.2 the plot of $k^{\frac{3}{4}}c_k$ is presented. The Baez-Duarte criterion requires this "wave" to be contained in the strip of parallel lines for all k. The violation of the RH would manifest as an increase of the amplitude of the combination $k^{\frac{3}{4}}c_k$ for large k. This point is elaborated in more detail by Maślanka in [6]. Here I will make some further comments on this issue. First it should be remarked that the r.h.s. of (16) consists of products of three terms: the first depending only on k (the overall factor $k^{\frac{3}{4}}$), the second depending only on imaginary parts of nontrivial zeros of ζ (the coefficients α_l and β_l) and third ingredients depending both on k and l (the trigonometric functions). Assume there are some zeros of ζ off critical line. We can split the sum over zeros ρ in (11) in two parts: one over zeros on critical line and

second over zeros off critical line. This second sum should violate the overall term $k^{-3/4}$ present in the first sum. Let $\gamma_i^{(o)}$ denote the imaginary parts of the zeros lying off critical line ("o" stands for "off"). It is not clear whether asymptotic similar to (19) will be valid for zeros off critical line, but it seems to be reasonable to assume that it should not differ significantly from (19). Then the contribution to c_k of such zeros off critical line should contain a factor of the order $e^{-\gamma_l^{(o)}}$. Because value of the imaginary part $\gamma_l^{(o)}$ of the hypothetical zero off critical line should be extremely large, perhaps even as large as 10^{100} can be expected, the combined contribution to c_k coming from the second sum seems to be extremely small, thus to see violation of the Baez-Duarte criterion the values of k should be larger than famous Skewes number and look something like $10^{10^{\cdot^{\cdot^{\cdot}}}}$. Such a big index k should cause that the first term in (16) overcame the smallness of the second term depending only on $\gamma_l^{(o)}$.

Table V

	extrapolated zero k'	γ_1 from eq. (25)
1	19319.0191151	0.66394362155812867
2	22526.0331312	40.91301517418643030
3	41868.9707418	10.13657613078084811
4	60094.3311655	17.38744852979003230
5	98378.2514809	12.74744116303923005
6	149320.1629630	15.05785832239914317
7	236817.0977574	13.62376898016027559
8	366000.4553802	14.43265811387491946
9	573460.6753253	13.99205468779138406
10	891841.3774543	14.22828231751450665
11	1393469.2943691	14.07955314904129825
12	2173554.0482344	14.13327453400666383
13	3387835.5708183	14.15682029348025265
14	5283842.8916393	14.13659885512530467
15	8247263.1465316	14.11229699162269810
16	12864372.4128156	14.13284964864925287
17	20052822.4883780	14.15424390183719722
18	31271608.9210745	14.14046804681939058
19	48805962.1935733	14.11501826908551253
20	76145459.4891092	14.12608973663948854
21	118689214.0783221	14.15568694594979885
22	185076299.0519995	14.14304477168657435
23	288851950.0033488	14.11488330063446422
24	450663149.7663923	14.12567920473796440
25	702517003.8056590	14.15292988626078933

The plot in Fig.2 is a perfect sine of one wavelength thus it gives visual justifi-

cation of the above statement that $\tilde{c}$ is determined in fact by the first zero γ_1. The same phenomenon was mentioned by Maślanka in [6].

4. Final remarks

The formula (16) together with a few coefficients α_l and β_l taken from Table IV allows to compute values of c_k for arbitrary large k. Other criteria for RH, like the value of the de Bruijn-Newman constant [17], are vulnerable to the Lehmer pairs of zeros of $\zeta(s)$. It is hard to see the reason for violation of the inequality $|c_k| <$ const $k^{-\frac{3}{4}}$. I have checked, that at the first Lehmer pair $\rho_{6709} = 0.5+7005.06286617i$ and $\rho_{6710} = 0.5 + 7005.1005646i$ the derivative has value $\zeta'(0.5 + 7005.06286617i) = 3.2229849698 + 0.74179951875i$ and similar value for second zero, thus there is no chance to get values of c_k violating (2) in this way. It seems to be an open problem how to connect the value of the largest k for which $|c_k| < \text{const}k^{-\frac{3}{4}}$ to the number of zeros lying on the critical line. Let us mention that for the Li's criterion [18] which states that if the numbers

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}(s^{n-1}\log\xi(s))|_{s=1} \tag{27}$$

fulfill $\lambda_n > 0$ for each n then RH is true it is known that if the first n Li's constants λ_n are positive then every zero ρ of $\zeta(s)$ with $|\Im\rho| < \sqrt{n}$ lies on the critical line $\Re\rho = \frac{1}{2}$ [19].

After a few months of computer experiments with c_k I believe Baez-Duarte sequence is one the most important and mysterious sequences in the whole mathematics.

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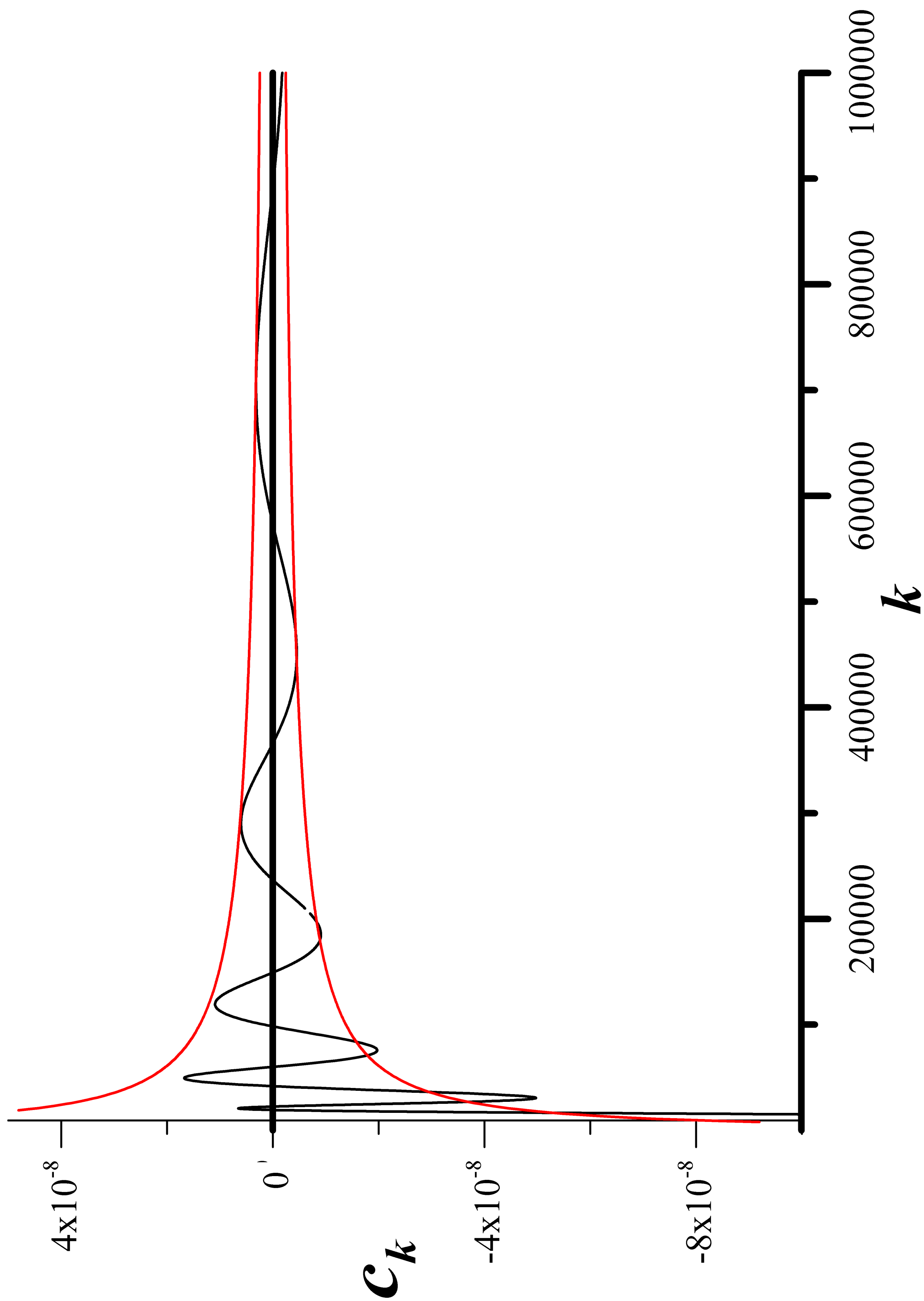

Fig. 1: The plot of c_k for $k \in (1, 10^6)$. At $k = 200000$ a small gap is visible to distinguish between c_k calculated from generic formula (1) and from explicit formulas presented in Sec. 3.

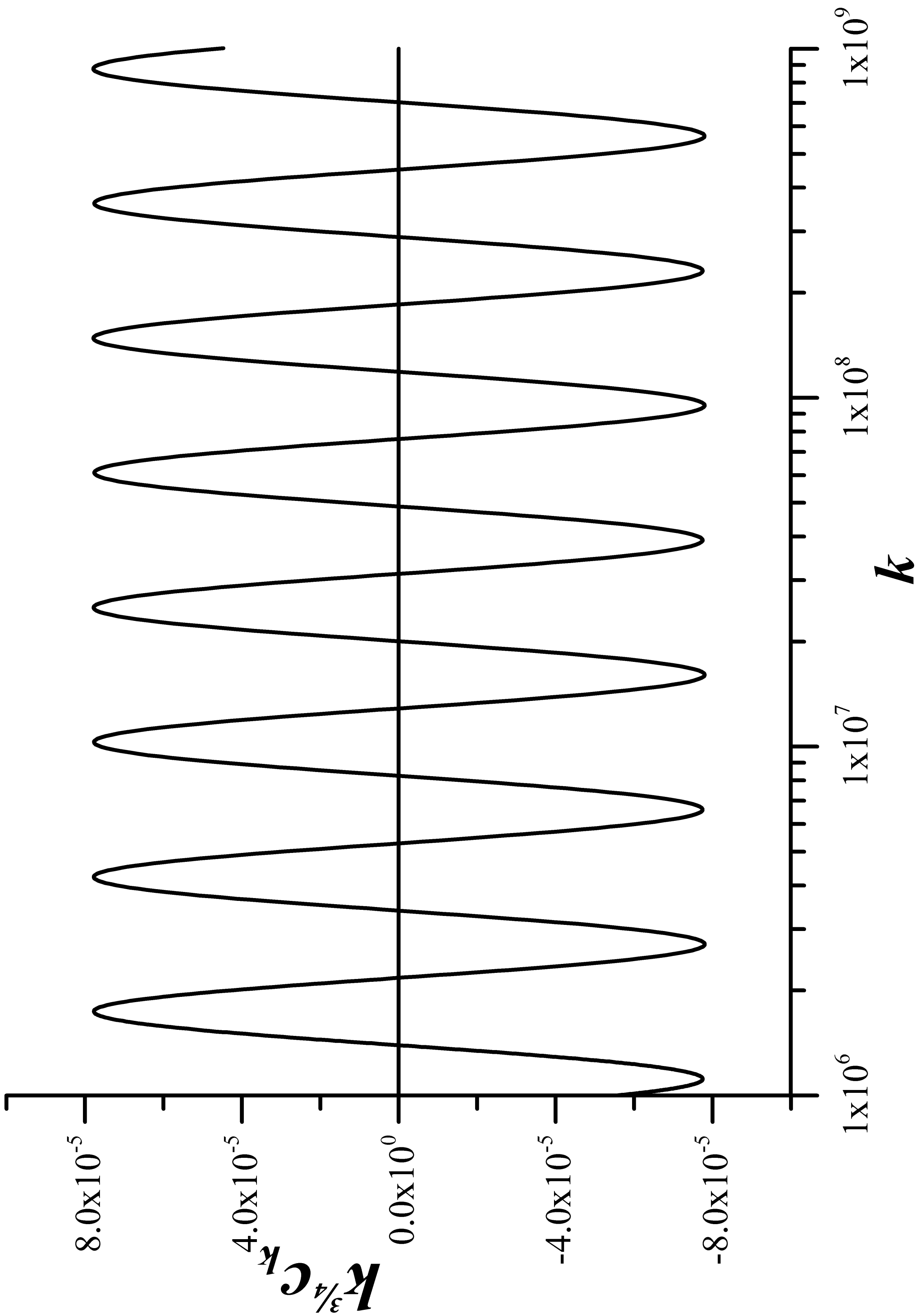

Fig. 2: The plot of $k^{3/4}c_k$ for $k \in (10^6, 10^9)$.